Proposition 1 *A static solution to equation (2), with $\delta > 0$ nodes inside the orbifold interval is always unstable.*

Proposition 2 *A static, nodeless solution $\phi_{A_*}(y)$ to equation (2), with amplitude A_*, and associated energy density $E(A_*)$ is stable if*

$$\left.\frac{dE}{dA}\right|_{A=A_*} < 0 \; . \tag{5}$$

This is a powerful result since it means that given any scalar potential $V(\phi)$ we immediately know which of the nontrivial nodeless solutions ϕ_A will be stable or unstable, without the need to actually know explicitly their analytic form.

With this result it is possible to understand the vacuum structure of any single scalar field theory with Dirichlet boundary conditions in 5D when the metric along the extra dimension is flat. Possible static solutions consist of the trivial solution $\langle\phi\rangle = 0$ (which may or may not be stable), kink-like solutions with nodes in the interval (which are always unstable), and kink-like solutions without nodes in the interval (some stable and some unstable, depending on condition (5)). As remarked in [40, 41], the trivial solution may be the true vacuum solution even in the case of a negative mass term $-|\mu^2|\phi^2$ in the 5D potential, as long as the inequality $|\mu^2| < |1/R^2|$ is preserved. Therefore, for a given orbifold radius R, many different perturbatively stable vacuum solutions are possible, and it is necessary to identify which one is the true vacuum of the theory.

The true vacuum of the theory will depend on the size of the radius R. This can be seen as follows: Without loss of generality, one may define the energy density of the trivial solution to be zero by choosing the 5D potential $V(\phi)$ to vanish at $\phi = 0$. It was shown in [40, 41] that there is a critical radius R_c below which nontrivial nodeless solutions do not exist (see Fig. 2). The energy density associated with the critical nontrivial nodeless solution will be either positive or exactly zero, so that the transition from one vacuum to another can be either second order or first order, as one varies the radius R.

In Fig. 3 we show an example of a simple setup defined by the scalar field potential $V(\phi) = -\frac{1}{2}|\mu^2|\phi^2 - \frac{1}{4}|\lambda|\phi^4 + \frac{1}{6}|\xi|\phi^6$, with $\mu^2 = 4M_*^2$, $\lambda = 4M_*^{-1}$ and $\xi = 0.6M_*^{-4}$. In the right panel, the energy density of two static solutions is plotted as a function of R, showing clearly that below a critical radius R_1 only the trivial solution is possible and above a critical radius R_2 only the kink solution is possible. For $R_1 < R < R_2$, both solutions